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NOTE ON THE GENERAL PROJECTIVE TRANSFORMATION.

By Dr. E. O. Lovett, Leipzig, Germany.

The general projective transformation in space of three dimensions is an operation that carries every plane into the position of some plane. This second position of the plane may coincide with the first position; the transformation is then the identical transformation. Two planes with their line of intersection will be transferred to the position of two planes with a corresponding line of intersection; hence every right line is carried into the position of some right line, i. e. the family of ∞^4 right lines in space is invariant by the general projective transformation. In general no individual right line is invariant, but the family as a whole is invariant.

It is proposed to make use of this property, namely, the invariance of the right line by the transformation, to determine the finite forms of the general projective transformation in ordinary space.*

Let the equations of the right line be those of the two planes:

$$E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0\,,$$

$$E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0\,;$$

then without loss of generality these may be taken in their equivalent forms:

$$\begin{aligned} y &= ax + b\,, \\ z &= cx + d\,. \end{aligned} \tag{1}$$

By differentiation and elimination these equations give

$$\frac{d^2y}{dx^2} \equiv y'' = 0\,, \quad \frac{d^2z}{dx^2} \equiv z'' = 0\,. \tag{2}$$

Then the ∞^4 right lines in space represented by the equations (1) are the integral-curves of the simultaneous differential equations of the second order (2). Hence these two differential equations (2) are themselves invariant by the transformation.†

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* The general form of the projective transformation of the plane is derived from the invariance of the right line in the plane in Lie's "Vorlesungen über continuierliche Gruppen," pp. 33 et seq.—Leipzig, 1893.

† According to the theory of invariant differential equations a system of simultaneous differential equations is said to be invariant under a given transformation when the geometrical representatives of the given system constitute an invariant family. Thus the criterion that a given system be invariant is that the transformed equations shall represent the same family of geometrical figures that the original equations represented.

Now let the general projective transformation be expressed analytically by

$$x_1 = \varphi(x, y, z), \quad y_1 = \psi(x, y, z), \quad z_1 = \omega(x, y, z). \tag{3}$$

Then the transformation (3) transforms right line into right line, when φ, ψ and ω are such functions of x, y, z that the differential equations (2) have the same forms in the new variables, namely,

$$\frac{d^2y_1}{dx_1^{\,2}} \equiv y_1'' = 0, \quad \frac{d^2z_1}{dx_1^{\,2}} \equiv z_1'' = 0, \tag{4}$$

as they had in the old variables x, y, z. This condition (4) furnishes a means of determining the forms of the functions φ, ψ, ω.

Equations (3) give by differentiation

$$\begin{aligned} \frac{dy_1}{dx_1} &= \frac{\psi_x + \psi_y y' + \psi_z z'}{\varphi_x + \varphi_y y' + \varphi_z z'}, \\ \frac{dz_1}{dx_1} &= \frac{\omega_x + \omega_y y' + \omega_z z'}{\varphi_x + \varphi_y y' + \varphi_z z'}. \end{aligned} \tag{5}$$

Further,

$$\begin{aligned} \frac{d^2y_1}{dx_1^{\,2}} = \frac{d\,\dfrac{dy_1}{dx_1}}{dx} : \frac{dx_1}{dx} = {} & \frac{1}{(\varphi_x + \varphi_y y' + \varphi_z z')^3}\{(\varphi_x + \varphi_y y' + \varphi_z z')(\psi_{xx} + 2\psi_{xy} y' \\ & + \psi_{yy} y'^2 + \psi_y y'' + 2\psi_{yz} y' z' + 2\psi_{zx} z' + \psi_{zz} z'^2 + \psi_z z'') - (\psi_x + \psi_y y' \\ & + \psi_z z')(\varphi_{xx} + 2\varphi_{xy} y' + \varphi_{yy} y'^2 + \varphi_y y'' + 2\varphi_{yz} y' z' + 2\varphi_{zx} z' + \varphi_{zz} z'^2 \\ & + \varphi_z z'')\} \\ = {} & N\{(\varphi_x\psi_{xx} - \psi_x\varphi_{xx}) + (\varphi_y\psi_{xx} + 2\varphi_x\psi_{xy} - \psi_y\varphi_{xx} - 2\psi_x\varphi_{xy})y' \\ & + (\varphi_z\psi_{xx} + 2\varphi_x\psi_{zx} - \psi_z\varphi_{xx} - 2\psi_x\varphi_{zx})z' \\ & + (\varphi_x\psi_{yy} + 2\varphi_y\psi_{xy} - \psi_x\varphi_{yy} - 2\psi_y\varphi_{xy})y'^2 \\ & + (\varphi_x\psi_{zz} + 2\varphi_z\psi_{zx} - \psi_x\varphi_{zz} - 2\psi_z\varphi_{zx})z'^2 \\ & + 2(\varphi_x\psi_{yz} + \varphi_y\psi_{zx} + \varphi_z\psi_{xy} - \psi_x\varphi_{yz} - \psi_y\varphi_{zx} - \psi_z\varphi_{xy})y'z' \\ & + (\varphi_y\psi_{yy} - \psi_y\varphi_{yy})y'^3 + (\varphi_z\psi_{zz} - \psi_z\varphi_{zz})z'^3 \\ & + (\varphi_y\psi_{zz} + 2\varphi_z\psi_{yz} - \psi_y\varphi_{zz} - 2\psi_z\varphi_{yz})y'z'^2 \\ & + (\varphi_z\psi_{yy} + 2\varphi_y\psi_{yz} - \psi_z\varphi_{yy} - 2\psi_y\varphi_{yz})z'y'^2\}, \end{aligned} \tag{6}$$

since $y'' = z'' = 0$, where $N = \dfrac{1}{(\varphi_x + \varphi_y y' + \varphi_z z')^3}$, $\varphi_x = \dfrac{\partial\varphi}{\partial x}$, $\varphi_{xx} = \dfrac{\partial^2\varphi}{\partial x^2}$, $y' = \dfrac{dy}{dx}$, etc.

Similarly

$$
\begin{aligned}
\frac{d^2z_1}{dx_1^2} = N\{&(\varphi_x\omega_{xx} - \omega_x\varphi_{xx}) + (\varphi_y\omega_{xx} + 2\varphi_x\omega_{xy} - \omega_y\varphi_{xx} - 2\omega_x\varphi_{xy})\,y' \\
&+ (\varphi_z\omega_{xx} + 2\varphi_x\omega_{zx} - \omega_z\varphi_{xx} - 2\omega_x\varphi_{zx})\,z' \\
&+ (\varphi_x\omega_{yy} + 2\varphi_y\omega_{xy} - \omega_x\varphi_{yy} - 2\omega_y\varphi_{xy})\,y'^2 \\
&+ (\varphi_x\omega_{zz} + 2\varphi_z\omega_{zx} - \omega_x\varphi_{zz} - 2\omega_z\varphi_{zx})\,z'^2 \\
&+ 2\,(\varphi_x\omega_{yz} + \varphi_y\omega_{zx} + \varphi_z\omega_{xy} - \omega_x\varphi_{yz} - \omega_y\varphi_{zx} - \omega_z\varphi_{xy})\,y'z' \\
&+ (\varphi_y\omega_{yy} - \omega_y\varphi_{yy})\,y'^3 + (\varphi_z\omega_{zz} - \omega_z\varphi_{zz})\,z'^3 \\
&+ (\varphi_y\omega_{zz} + 2\varphi_z\omega_{yz} - \omega_y\varphi_{zz} - 2\omega_z\varphi_{yz})\,y'z'^2 \\
&+ (\varphi_z\omega_{yy} + 2\varphi_y\omega_{yz} - \omega_z\varphi_{yy} - 2\omega_y\varphi_{yz})\,z'y'^2\}\,. \qquad (7)
\end{aligned}
$$

These expressions (6) and (7) must be equal to zero for all values of x, y, z, y', z' by virtue of $y_1'' = 0$ and $z_1'' = 0$. Therefore, the conditions that y'' and z'' be invariant are given by the following system of partial differential equations :

$$
\left.
\begin{aligned}
\varphi_x\psi_{xx} - \psi_x\varphi_{xx} &= 0\,, \\
\varphi_x\omega_{xx} - \omega_x\varphi_{xx} &= 0\,, \\
\varphi_y\psi_{yy} - \psi_y\varphi_{yy} &= 0\,, \\
\varphi_y\omega_{yy} - \omega_y\varphi_{yy} &= 0\,, \\
\varphi_z\psi_{zz} - \psi_z\varphi_{zz} &= 0\,, \\
\varphi_z\omega_{zz} - \omega_z\varphi_{zz} &= 0\,;
\end{aligned}
\right\} \qquad (8)
$$

$$
\left.
\begin{aligned}
\varphi_y\psi_{xx} - \psi_y\varphi_{xx} + 2\,(\varphi_x\psi_{xy} - \psi_x\varphi_{xy}) &= 0\,, \\
\varphi_y\omega_{xx} - \omega_y\varphi_{xx} + 2\,(\varphi_x\omega_{xy} - \omega_x\varphi_{xy}) &= 0\,, \\
\varphi_z\psi_{xx} - \psi_z\varphi_{xx} + 2\,(\varphi_x\psi_{zx} - \psi_x\varphi_{zx}) &= 0\,, \\
\varphi_z\omega_{xx} - \omega_z\varphi_{xx} + 2\,(\varphi_x\omega_{zx} - \omega_x\varphi_{zx}) &= 0\,, \\
\varphi_x\psi_{yy} - \psi_x\varphi_{yy} + 2\,(\varphi_y\psi_{xy} - \psi_y\varphi_{xy}) &= 0\,, \\
\varphi_x\omega_{yy} - \omega_x\varphi_{yy} + 2\,(\varphi_y\omega_{xy} - \omega_y\varphi_{xy}) &= 0\,, \\
\varphi_x\psi_{zz} - \psi_x\varphi_{zz} + 2\,(\varphi_z\psi_{zx} - \psi_z\varphi_{zx}) &= 0\,, \\
\varphi_x\omega_{zz} - \omega_x\varphi_{zz} + 2\,(\varphi_z\omega_{zx} - \omega_z\varphi_{zx}) &= 0\,;
\end{aligned}
\right\} \qquad (9)
$$

$$\left.\begin{aligned}
\varphi_y\psi_{zz} - \psi_y\varphi_{zz} + 2(\varphi_z\psi_{yz} - \psi_z\varphi_{yz}) &= 0\,,\\
\varphi_y\omega_{zz} - \omega_y\varphi_{zz} + 2(\varphi_z\omega_{yz} - \omega_z\varphi_{yz}) &= 0\,,\\
\varphi_z\psi_{yy} - \psi_z\varphi_{yy} + 2(\varphi_y\psi_{yz} - \psi_y\varphi_{yz}) &= 0\,,\\
\varphi_z\omega_{yy} - \omega_z\varphi_{yy} + 2(\varphi_y\omega_{yz} - \omega_y\varphi_{yz}) &= 0\,,\\
\varphi_x\psi_{yz} - \psi_x\varphi_{yz} + \varphi_y\psi_{zx} - \psi_y\varphi_{zx} + \varphi_z\psi_{xy} - \psi_z\varphi_{xy} &= 0\,,\\
\varphi_x\omega_{yz} - \omega_x\varphi_{yz} + \varphi_y\omega_{zx} - \omega_y\varphi_{zx} + \varphi_z\omega_{xy} - \omega_z\varphi_{xy} &= 0\,.
\end{aligned}\right\}\qquad(10)$$

Hence, in order that the equations (4) may represent the general projective transformation in three variables it is necessary and sufficient that the functions φ, ψ and ω satisfy the system of partial differential equations (8), (9), (10). Then the integration of this system will give the forms of φ, ψ and ω.

From the equations (8) it follows that

$$\begin{aligned}
\frac{\partial \log \psi_x}{\partial x} &= \frac{\partial \log \varphi_x}{\partial x} = \frac{\partial \log \omega_x}{\partial x}\,,\\
\frac{\partial \log \psi_y}{\partial y} &= \frac{\partial \log \varphi_y}{\partial y} = \frac{\partial \log \omega_y}{\partial y}\,,\\
\frac{\partial \log \psi_z}{\partial z} &= \frac{\partial \log \varphi_z}{\partial z} = \frac{\partial \log \omega_z}{\partial z}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi_x &= P(y, z)\,.\,\varphi_x\,, & \psi_y &= Q(z, x)\,.\,\varphi_y\,, & \psi_z &= R(x, y)\,.\,\varphi_z\,,\\
\omega_x &= S(y, z)\,.\,\varphi_x\,, & \omega_y &= T(z, x)\,.\,\varphi_y\,, & \omega_z &= U(x, y)\,.\,\varphi_z\,,
\end{aligned}\qquad(11)$$

in which P and S are functions of y and z alone; Q and T of z and x alone; R and U of x and y alone.

By substituting these values in equations (9) the latter become

$$\begin{aligned}
(P - Q)\,\varphi_{xx}\varphi_y + 2P_y\varphi_x^2 &= 0\,,\\
(P - Q)\,\varphi_{yy}\varphi_x - 2Q_x\varphi_y^2 &= 0\,,\\
(P - R)\,\varphi_{xx}\varphi_z + 2P_z\varphi_x^2 &= 0\,,\\
(P - R)\,\varphi_{zz}\varphi_x - 2R_x\varphi_z^2 &= 0\,,\\
(S - T)\,\varphi_{xx}\varphi_y + 2S_y\varphi_x^2 &= 0\,,\\
(S - T)\,\varphi_{yy}\varphi_x - 2T_x\varphi_y^2 &= 0\,,\\
(S - U)\,\varphi_{xx}\varphi_z + 2U_z\varphi_x^2 &= 0\,,\\
(S - U)\,\varphi_{zz}\varphi_x - 2U_x\varphi_z^2 &= 0\,.
\end{aligned}\qquad(12)$$

Further, since $\psi_{xy} = \psi_{yx}$, $\psi_{xz} = \psi_{zx}$, $\omega_{xy} = \omega_{yx}$, . . . , equations (11) give

$$\left.\begin{aligned}
(P - Q)\,\varphi_{xy} + P_y\varphi_x - Q_x\varphi_y &= 0\,,\\
(P - R)\,\varphi_{xz} + P_z\varphi_x - R_x\varphi_z &= 0\,,\\
(Q - R)\,\varphi_{yz} + Q_z\varphi_y - R_y\varphi_z &= 0\,,\\
(S - T)\,\varphi_{xy} + S_y\varphi_x - T_x\varphi_y &= 0\,,\\
(S - U)\,\varphi_{xz} + S_z\varphi_x - U_x\varphi_z &= 0\,,\\
(T - U)\,\varphi_{yz} + T_z\varphi_y - U_y\varphi_z &= 0\,.
\end{aligned}\right\} \tag{13}$$

By differentiation of the equations (12) partially with respect to x, y, x, z, x, y, x, z, respectively, we have the following relations :

$$Q_x\varphi_{xx}\varphi_y - (P - Q)\,(\varphi_{xxx}\varphi_y + \varphi_{xx}\varphi_{xy}) - 4P_y\varphi_x\varphi_{xx} = 0\,, \tag{14}$$

$$P_y\varphi_{yy}\varphi_x + (P - Q)\,(\varphi_{yyy}\varphi_x + \varphi_{yy}\varphi_{xy}) - 4Q_x\varphi_y\varphi_{yy} = 0\,, \tag{15}$$

$$R_x\varphi_{xx}\varphi_z - (P - R)\,(\varphi_{xxx}\varphi_z + \varphi_{xx}\varphi_{xz}) - 4P_z\varphi_x\varphi_{xx} = 0\,, \tag{16}$$

$$P_z\varphi_{zz}\varphi_x + (P - R)\,(\varphi_{zzz}\varphi_x + \varphi_{zz}\varphi_{xz}) - 4R_x\varphi_z\varphi_{zz} = 0\,; \tag{17}$$

together with four similar equations in which S, T, U, take the place of P, Q, R, respectively.

Equations (12) and (13) give the following:

$$\varphi_{xx} = \frac{-\,2P_y\varphi_x^2}{(P - Q)\,\varphi_y}\,, \tag{18}$$

$$\varphi_{xy} = \frac{Q_x\varphi_y - P_y\varphi_x}{P - Q}\,, \tag{19}$$

$$\varphi_{yy} = \frac{2Q_x\varphi_y^2}{(P - Q)\,\varphi_x}\,, \tag{20}$$

$$\varphi_{xz} = \frac{R_x\varphi_z - P_z\varphi_x}{P - R}\,, \tag{21}$$

$$\varphi_{zz} = \frac{2R_x\varphi_z^2}{(P - R)\,\varphi_x}\,. \tag{22}$$

Substituting (18) and (19) in (14) :

$$\begin{aligned}
&\frac{Q_x\varphi_y\,.\,2P_y\varphi_x^2}{(P - Q)\,\varphi_y} + (P - Q)\,\varphi_{xxx}\varphi_y\\
&\quad - \frac{2P_y\varphi_x^2\,(Q_x\varphi_y - P_y\varphi_x)}{(P - Q)\,\varphi_y} - \frac{8P_y\varphi_xP_y\varphi_x^2}{(P - Q)\,\varphi_y} = 0\,,
\end{aligned} \tag{23}$$

Substituting (19) and (20) in (15),

$$\frac{P_y\varphi_x \cdot 2Q_x\varphi_y^2}{(P-Q)\varphi_x} + (P-Q)\,\varphi_{yyy}\varphi_x + \frac{2Q_x\varphi_y^2\,(Q_x\varphi_y - P_y\varphi_x)}{(P-Q)\,\varphi_x} - \frac{8Q_x\varphi_yQ_x\varphi_y^2}{(P-Q)\,\varphi_x} = 0\,, \tag{24}$$

Substituting (21) and (22) in (17),

$$\frac{P_z\varphi_x \cdot 2R_x\varphi_z^2}{(P-R)\,\varphi_x} + (P-R)\,\varphi_{zzz}\varphi_x + \frac{2R_x\varphi_z^2\,(R_x\varphi_z - P_z\varphi_x)}{(P-R)\,\varphi_x} - \frac{8R_x\varphi_z \cdot R_x\varphi_z^2}{(P-R)\,\varphi_x} = 0\,, \tag{25}$$

These equations become, after obvious reductions,

$$\varphi_{xxx} = \frac{6P_y^2}{(P-Q)^2}\,\frac{\varphi_x^3}{\varphi_y^2}\,, \tag{26}$$

$$\varphi_{yyy} = \frac{6Q_x^2}{(P-Q)^2}\,\frac{\varphi_y^3}{\varphi_x^2}\,, \tag{27}$$

$$\varphi_{zzz} = \frac{6R_x^2}{(P-R)^2}\cdot\frac{\varphi_z^3}{\varphi_x^2}\,. \tag{28}$$

Solving the first, second, and fourth equations of (12) for $P_y/(P-Q)$, $Q_x/(P-Q)$, $R_x/(P-R)$, respectively, and substituting in the last three equations, they become

$$\frac{\varphi_{xxx}}{\varphi_{xx}} = \frac{3}{2}\,\frac{\varphi_{xx}}{\varphi_x}\,, \quad \frac{\varphi_{yyy}}{\varphi_{yy}} = \frac{3}{2}\,\frac{\varphi_{yy}}{\varphi_y}\,, \quad \frac{\varphi_{zzz}}{\varphi_{zz}} = \frac{3}{2}\,\frac{\varphi_{zz}}{\varphi_z}\,. \tag{29}$$

Integrating these equations we have, $\frac{\varphi_{xx}^2}{\varphi_x^3}$ a function of y and z alone ; $\frac{\varphi_{yy}^2}{\varphi_y^3}$ a function of x and z alone ; $\frac{\varphi_{zz}^2}{\varphi_z^3}$ a function of x and y alone. The same is obviously true of $\frac{\varphi_{xx}}{\varphi_x^{\frac{3}{2}}}$, $\frac{\varphi_{yy}}{\varphi_y^{\frac{3}{2}}}$, $\frac{\varphi_{zz}}{\varphi_z^{\frac{3}{2}}}$; therefore the integrals of these $\frac{1}{\varphi_x^{\frac{1}{2}}}$, $\frac{1}{\varphi_y^{\frac{1}{2}}}$, $\frac{1}{\varphi_z^{\frac{1}{2}}}$ are linear in x, y, and z respectively ; and therefore φ_x, φ_y, φ_z have the forms,

$$\varphi_x = \frac{1}{(\alpha x + \beta)^2}\,, \quad \varphi_y = \frac{1}{(\gamma y + \delta)^2}\,, \quad \varphi_z = \frac{1}{(\varepsilon z + \zeta)^2}\,; \tag{30}$$

in which α and β are functions of y and z ; γ and δ are functions of x and z ; ε and ζ are functions of x and y.

These give φ the forms,

$$\frac{1}{\alpha x+\beta}+\eta(y,z)\,,\ \text{or}\ \varphi=\frac{\kappa x+\nu}{\alpha x+\beta};$$

$$\frac{1}{\gamma y+\delta}+\theta(x,z)\,,\ \text{or}\ \varphi=\frac{\lambda y+o}{\gamma y+\delta};$$

$$\frac{1}{\varepsilon z+\zeta}+\iota(x,y)\,,\ \text{or}\ \varphi=\frac{\mu z+\pi}{\varepsilon z+\zeta};$$

where κ, ν, α, β are functions of y and z; λ, o, γ, δ are functions of x and z; μ, π, ε, ζ are functions of x and y.

Therefore φ has the form of a fraction whose numerator and denominator are linear in x, y, and z, that is

$$\varphi=\frac{a_1x+b_1y+c_1z+d_1+e_1yz+f_1zx+g_1xy+h_1xyz}{a_4x+b_4y+c_4z+d_4+e_4yz+f_4zx+g_4xy+h_4xyz}; \tag{31}$$

similarly

$$\psi=\frac{a_2x+b_2y+c_2z+d_2+e_2yz+f_2zx+g_2xy+h_2xyz}{a_4x+b_4y+c_4z+d_4+e_4yz+f_4zx+g_4xy+h_4xyz}, \tag{32}$$

and

$$\omega=\frac{a_3x+b_3y+c_3z+d_3+e_3yz+f_3zx+g_3xy+h_3xyz}{a_4x+b_4y+c_4z+d_4+e_4yz+f_4zx+g_4xy+h_4xyz}. \tag{33}$$

It will be noticed that the denominators of φ, ψ and ω are the same, this follows from equations (11). For, supposing

$$\varphi=\frac{D(x,y,z)}{L(x,y,z)},\ \text{and}\ \psi=\frac{E(x,y,z)}{M(x,y,z)}, \tag{34}$$

where D, E, L, M are functions linear in x, y, and z, then the equations (11) give

$$\frac{\psi_x}{\varphi_x}=\frac{L^2}{M^2}\frac{(ME_x-EM_x)}{(LD_x-DL_x)}=P(y,z); \tag{35}$$

$$\frac{\psi_y}{\varphi_y}=\frac{L^2}{M^2}\frac{(ME_y-EM_y)}{(LD_y-DL_y)}=Q(z,x); \tag{36}$$

$$\frac{\psi_z}{\varphi_z}=\frac{L^2}{M^2}\frac{(ME_z-EM_z)}{(LD_z-DL_z)}=R(x,y). \tag{37}$$

But, since D, E, L, and M are linear in x, y, and z, the parentheses in (35), (36), (37) are themselves functions, respectively, of y, z; x, z; x, y. Therefore (35) says that L and M are the same as far as x is concerned, (36) that they have the same form relative to y, and (37) that z enters both in the same manner ; therefore L and M have the same form and can differ only by a constant factor. But a constant factor will not change the nature of the transformation, since it can only change the coefficients and all of them in the same ratio.

If now the forms (34), (35), (36) be substituted in the equations of condition (9) and (10) it will be seen that we must have $e_1 = f_1 = g_1 = h_1 = e_2 = f_2 = g_2 = h_2 = e_3 = f_3 = g_3 = h_3 = e_4 = f_4 = g_4 = h_4 = 0$, in order that (9) and (10) may be satisfied.

Therefore, finally, φ, ψ and ω have the forms :

$$x_1 = \varphi(x, y, z) \equiv \frac{a_1x + b_1y + c_1z + d_1}{a_4x + b_4y + c_4z + d_4}, \tag{38}$$

$$y_1 = \psi(x, y, z) \equiv \frac{a_2x + b_2y + c_2z + d_2}{a_4x + b_4y + c_4z + d_4}, \tag{39}$$

$$z_1 = \omega(x, y, z) \equiv \frac{a_3x + b_3y + c_3z + d_3}{a_4x + b_4y + c_4z + d_4}, \tag{40}$$

as the analytical expressions of the general projective transformation in ordinary space.

It is possible to determine the finite and infinitesimal forms of the transformation directly from its definition, namely, that it leaves the plane invariant, or more strictly the family of ∞^3 planes in space is invariant by the transformation. Thus φ, ψ, and ω might be found from the conditions that the equations $\frac{\partial^2 t}{\partial x^2} = 0$, $\frac{\partial^2 t}{\partial x \partial y} = 0$, $\frac{\partial^2 t}{\partial y^2} = 0$, the differential equations of the plane, preserve their form in the new variables x_1, y_1, z_1, but the process is much more complicated than the one used above. The method here employed may be extended to the case of n variables and the determination of the finite forms of the general projective transformation in space of n dimensions.

As before we have

$$E_j \equiv a_{j,1}x_1 + a_{j,2}x_2 + \dots + a_{j,n}x_n = 0, \quad j = 1, \dots, n-1.$$

This is a system of $(n-1)$ equations in n variables $x_1, x_2, \dots, x_n$ and hence may be written in the form :

$$x_k = a_k x_1 + b_k. \qquad k = 2, \dots, n$$

Then

$$\frac{d^2 x_k}{dx_1^2} = x_k'' = 0. \qquad k = 2, \dots, n$$

Let the transformation be represented analytically by

$$\bar{x}_i = \varphi_i(x_1, x_2, \dots, x_n). \qquad i = 1, \dots, n.$$

Then the condition of the invariance of the right line is given by

$$\bar{x}_k'' = 0 \qquad\qquad k = 2, \ldots, n.$$

for all values of x_i and x_i'. Hence the conditions :

$$\begin{aligned} \varphi_{1,x_i}\varphi_{j,x_ix_i} - \varphi_{j,x_i}\varphi_{1,x_ix_i} &= 0, \\ \varphi_{1,x_i}\varphi_{j,x_kx_k} + 2\varphi_{1,x_k}\varphi_{j,x_ix_k} - \varphi_{j,x_i}\varphi_{1,x_kx_k} - 2\varphi_{j,x_k}\varphi_{1,x_ix_k} &= 0, \\ \varphi_{1,x_i}\varphi_{j,x_kx_l} + \varphi_{1,x_k}\varphi_{j,x_ix_l} + \varphi_{1,x_l}\varphi_{j,x_ix_k} & \\ - \varphi_{j,x_i}\varphi_{1,x_kx_l} - \varphi_{j,x_k}\varphi_{1,x_ix_l} - \varphi_{j,x_l}\varphi_{1,x_ix_k} &= 0. \end{aligned} \tag{41}$$

$$i, j, k, l = 1, \ldots, n.$$

And, as in the preceding case,

$$\bar{x}_i = \varphi_i(x_1, x_2 \ldots, x_n) = \frac{a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n + a_{i,n+1}}{a_1x_1 + a_2x_2 + \ldots + a_nx_n + a_{n+1}} \tag{42}$$

will be found to satisfy the preceding system of partial differential equations. Geometrically the relations (42) represent the general projective point-transformation in space of n-dimensions. Analytically they assert that n functions φ_i of the variables x_i subject to the conditions (41) have the form (42).

The forms of the infinitesimal projective transformation* may be readily derived from the finite forms (38), (39), and (40). An infinitesimal point transformation is an operation by which a point is carried into the position of some point at an infinitesimal distance from the original position. Two transformations are inverse the one of the other when a point finds itself in its original position after the successive application of the two transformations, i. e. a transformation followed by its inverse gives the identical transformation.

Solving (38), (39), and (40) for x, y, and z, we have

$$x = \frac{a_1'x_1 + b_1'y_1 + c_1'z_1 + d_1'}{a_4'x_1 + b_4'y_1 + c_4'z_1 + d_4'}, \tag{43}$$

$$y = \frac{a_2'x_1 + b_2'y_1 + c_2'z_1 + d_2'}{a_4'x_1 + b_4'y_1 + c_4'z_1 + d_4'}, \tag{44}$$

$$z = \frac{a_3'x_1 + b_3'y_1 + c_3'z_1 + d_3'}{a_4'x_1 + b_4'y_1 + c_4'z_1 + d_4'}. \tag{45}$$

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* The notion of a projective transformation dates back as far as Apollonius. The designation "projective" appears in Poncelet's "Traité des proprietés projectives des figures" 1822. Möbius gave the first analytical representation of the projective transformation, and in homogeneous coordinates, in "Der barycentrische Calcul" 1827. But the idea and theory of infinitesimal transformations, of which the infinitesimal projective transformation is but one type, are due to Lie.

The transformation (43), (44), (45) is the inverse of that represented by (38), (39) and (40), and has exactly the same form as the latter. Then the group of all projective transformations in ordinary space contains the inverse of every one of its transformations. The successive application of the above two transformations then gives

$$x_1 = x\,, \quad y_1 = y\,, \quad z_1 = z\,, \tag{46}$$

the identical transformation. It follows, therefore, that by proper choice of the constants (38), (39), and (40) should reduce to (46). This choice of constants is obviously expressed by the system of values below :

$$\begin{array}{llll} a_1 = 1\,, & b_1 = 0\,, & c_1 = 0\,, & d_1 = 0\,; \\ a_2 = 0\,, & b_2 = 1\,, & c_2 = 0\,, & d_2 = 0\,; \\ a_3 = 0\,, & b_3 = 0\,, & c_3 = 1\,, & d_3 = 0\,; \\ a_4 = 0\,, & b_4 = 0\,, & c_4 = 0\,, & d_4 = 1\,. \end{array}$$

Hence, since this system gives the identical transformation, to obtain the infinitesimal transformation* it is only necessary to set in (38), (39), and (40) values differing by an infinitesimal from the above. Then, if δt be an infinitesimal, and α_1, α_2, . . . , arbitrary constants, the following are the values of the constants in the infinitesimal point transformation in three variables :

$$\begin{array}{llll} a_1 = 1 + \alpha_1\delta t\,, & b_1 = \beta_1\delta t\,, & c_1 = \gamma_1\delta t\,, & d_1 = \delta_1\delta t\,; \\ a_2 = \alpha_2\delta t\,, & b_2 = 1 + \beta_2\delta t\,, & c_2 = \gamma_2\delta t\,, & d_2 = \delta_2\delta t\,; \\ a_3 = \alpha_3\delta t\,, & b_3 = \beta_3\delta t\,, & c_3 = 1 + \gamma_3\delta t\,, & d_3 = \delta_3\delta t\,; \\ a_4 = \alpha_4\delta t\,, & b_4 = \beta_4\delta t\,, & c_4 = \gamma_4\delta t\,, & d_4 = 1 + \delta_4\delta t\,. \end{array}$$

By substituting these values in (38), (39), and (40), they give the following general expressions for the infinitesimal projective transformation in space :

$$\begin{aligned} x_1 &= \frac{x + (\alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1)\,\delta t}{1 + (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t}\,, \\ y_1 &= \frac{y + (\alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2)\,\delta t}{1 + (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t}\,, \\ z_1 &= \frac{z + (\alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3)\,\delta t}{1 + (\alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4)\,\delta t}\,. \end{aligned} \tag{47}$$

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* See Page's article on "Transformation Groups," Annals of Mathematics, Vol. VIII, No. 4, p. 121.

But, as far as terms of the first order in δt,

$$\frac{1}{1+(\alpha_4 x+\beta_4 y+\gamma_4 z+\delta_4)\,\delta t}=1-(\alpha_4 x+\beta_4 y+\gamma_4 z+\delta_4)\,\delta t\,.$$

Therefore, as far as terms of the first order in δt,

$$\begin{aligned}
x_1&=x-(\alpha_4 x+\beta_4 y+\gamma_4 z+\delta_4)\,x\delta t+(\alpha_1 x+\beta_1 y+\gamma_1 z+\delta_1)\,\delta t\,,\\
y_1&=y-(\alpha_4 x+\beta_4 y+\gamma_4 z+\delta_4)\,y\delta t+(\alpha_2 x+\beta_2 y+\gamma_2 z+\delta_2)\,\delta t\,,\\
z_1&=z-(\alpha_4 x+\beta_4 y+\gamma_4 z+\delta_4)\,z\delta t+(\alpha_3 x+\beta_3 y+\gamma_3 z+\delta_3)\,\delta t\,.
\end{aligned}\tag{48}$$

Or,

$$\begin{aligned}
x_1&=x+\{\delta_1+(\alpha_1-\delta_4)\,x+\beta_1 y+\gamma_1 z-\alpha_4 x^2-\beta_4 xy-\gamma_4 zx\}\,\delta t\,,\\
y_1&=y+\{\delta_2+\alpha_2 x+(\beta_2-\delta_4)\,y+\gamma_2 z-\alpha_4 xy-\beta_4 y^2-\gamma_4 yz\}\,\delta t\,,\\
z_1&=z+\{\delta_3+\alpha_3 x+\beta_3 y+(\gamma_3-\delta_4)\,z-\alpha_4 xz-\beta_4 yz-\gamma_4 z^2\}\,\delta t\,;
\end{aligned}\tag{49}$$

where α, β, γ, δ are arbitrary constants and δt an infinitesimal.

By taking the following values of the constants :

$$\begin{array}{lllll}
\delta_1=a\,, & \alpha_1-\delta_4=e\,, & \beta_1=h\,, & \gamma_1=k\,, & -\alpha_4=n\,,\\
\delta_2=b\,, & \alpha_2=f\,, & \beta_2-\delta_4=i\,, & \gamma_2=l\,, & -\beta_4=p\,,\\
\delta_3=c\,, & \alpha_3=g\,, & \beta_3=j\,, & \gamma_3-\delta_4=m\,, & -\gamma_4=q\,,
\end{array}$$

the relations (40) take the following convenient forms :

$$\begin{aligned}
x_1&=x+(a+ex+hy+kz+nx^2+pxy+qzx)\,\delta t\,,\\
y_1&=y+(b+fx+iy+lz+nxy+py^2+qyz)\,\delta t\,,\\
z_1&=z+(c+gx+jy+mz+nxz+pyz+qz^2)\,\delta t\,,
\end{aligned}\tag{50}$$

the final expressions of the infinitesimal projective transformation in ordinary space. By it the coordinates x, y, z receive the increments,

$$\begin{aligned}
\delta x&=x_1-x=(a+ex+hy+kz+nx^2+pxy+qzx)\,\delta t\,,\\
\delta y&=y_1-y=(b+fx+iy+lz+nxy+py^2+qyz)\,\delta t\,,\\
\delta z&=z_1-z=(c+gx+jy+mz+nxz+pyz+qz^2)\,\delta t\,.
\end{aligned}\tag{51}$$

Any arbitrary function $f(x, y, z)$ will receive the increment

$$\delta f=\frac{\partial f}{\partial x}\,\delta x+\frac{\partial f}{\partial y}\,\delta y+\frac{\partial f}{\partial z}\,\delta z$$

by the transformation ; or

$$\frac{\delta f}{\delta t}=\frac{\partial f}{\partial x}\frac{\delta x}{\delta t}+\frac{\partial f}{\partial y}\frac{\delta y}{\delta t}+\frac{\partial f}{\partial z}\frac{\delta z}{\delta t}.$$

Now putting $\frac{\delta f}{\delta t}=Uf$ as a symbol of the operation, we have

$$\begin{aligned} Uf\equiv\ & (a+ex+hy+kz+nx^2+pxy+qzx)\frac{\partial f}{\partial x}\\ &+(b+fx+iy+lz+nxy+py^2+qyz)\frac{\partial f}{\partial y}\\ &+(c+gx+jy+mz+nxz+pyz+qz^2)\frac{\partial f}{\partial z}, \end{aligned}$$

as the general symbol of the general infinitesimal transformation in ordinary space, and this symbol defines the transformation completely. It is also noted that there are ∞^{14} infinitesimal projective transformations in ordinary space. Uf above contains fifteen arbitrary constants, but since δf is an arbitrary infinitesimal we may divide the expression for Uf through by one of the constants that is different from zero without altering the transformation ; hence there are only ∞^{14} infinitesimal projective transformations in ordinary space.

LEIPZIG, *November* 1, 1895.